

ON EIGENVALUE GENERIC PROPERTIES OF THE LAPLACE-NEUMANN OPERATOR

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ABSTRACT. In this work we establish the existence of analytic curves of eigenvalues for the Laplace-Neumann operator through an analytic variation of the metric of a compact Riemannian manifold M with boundary. Further, we also obtain an expression for the derivative of the curve of eigenvalues, which is used as a device to prove that the eigenvalues of Laplace-Neumann operator are generically simple in the space \mathcal{M}^k of C^k -metrics on M .

1. INTRODUCTION

In her seminal work [15] Uhlenbeck proved groundbreaking results on generic properties for eigenvalues and eigenfunctions of the Laplace-Beltrami operator Δ_g on a closed n -dimensional Riemannian manifold (M, g) . From a qualitative point of view, one of the most beautiful results in [15] is the celebrated Theorem 8 asserting that the set of the C^k -metrics g for which Δ_g has simple spectrum is residual in \mathcal{M}^k , for any $2 \leq k < \infty$. Over the last four decades, similar results were obtained in various directions, see for example [3, 4, 8, 10, 11, 13, 17].

In line with this theme, Micheletti and Pistoia [9] have proposed a sufficient condition for the set of the deformations of a Riemannian metric g , which preserve the multiplicity $m \geq 2$ of a fixed eigenvalue $\lambda(g)$ associated with g , to be locally a manifold of codimension $\frac{1}{2}m(m+1) - 1$ inside the Banach space $\mathcal{S}^{2,k}$ of all C^k symmetric covariant 2-tensors on M . They showed further that such a condition is easily fulfilled when $n = 2$ and $m = 2$ (cf. [9, Theorem 4.3]).

Nevertheless very little work has been done so far to address issues about generic properties of eigenvalues and eigenfunctions of the Laplace-Beltrami operator on compact Riemannian manifolds, subjected to boundary Neumann conditions. To shorten notation, it hereinafter will refer as Laplace-Neumann operator. The single and most significant reason for this, is due to the difficulty of dealing with normal derivatives of functions in boundary when g varies through \mathcal{M}^k . Furthermore, even with the help of important results from perturbation theory of linear operators, usually available for the Dirichlet case, the Laplace-Neumann operator requires own treatment.

The main goal of this work is to establish similar results to those cited above for the Laplace-Neumann operator on compact manifolds. Following [9] we let \mathcal{M}^k denote the set of all C^k Riemannian metrics on M . It follows that every metric $g \in \mathcal{M}^k$ determines a sequence

$$0 = \mu_0(g) < \mu_1(g) \leq \mu_2(g) \leq \cdots \leq \mu_k(g) \leq \cdots$$

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of eigenvalues of Δ_g counted with their multiplicities. We regard each eigenvalue $\mu_k(g)$ as a function of g in \mathcal{M}^k .

The crucial step in our approach is ensuring the existence of eigenvalue-curves given by a one-parameter family of Riemannian structures $g(t)$ on M . As is well-known, these curves were already obtained in the Dirichlet case by Berger [2] using the perturbation theory for linear operators of Kato [6]. However, as we will see in Section 4, the method employed in [2] cannot be applied to the Neumann-problem case. In order to overcome this drawback, our strategy focuses on reducing the infinite-dimensional problem to a finite-dimensional one and then use the Kato Selection theorem, as an optional device. To reduce the dimension we shall use the method of Liapunov-Schmidt, along the same lines of thinking as in [8]. After all, we namely prove the following theorem.

Theorem 1.1. *Let (M, g_0) be a compact oriented Riemannian manifold and $\{g(t)\}$ a real analytic one-parameter family of Riemannian structures on M with $g(0) = g_0$. Assume λ is an eigenvalue of multiplicity m for the Laplace-Neumann operator Δ_{g_0} . Then there exist $\varepsilon > 0$ and t -analytic functions $\lambda_i(t)$ and $\phi_i(t)$, ($i = 1, \dots, m$) such that $\langle \phi_i(t), \phi_j(t) \rangle_{L^2(M, \text{dvol}_{g(t)})} = \delta_i^j$ and the following two relations hold for every $|t| < \varepsilon$:*

- (i) $\Delta_{g(t)}\phi_i(t) = \lambda_i(t)\phi_i(t)$ in M ,
- (ii) $\frac{\partial}{\partial \nu_t}\phi_i(t) = 0$ on ∂M ,
- (iii) $\lambda_i(0) = \lambda$.

It is worthwhile to highlight once again that Berger's approach for proving the existence result above for the corresponding Laplace-Beltrami operator, using perturbation theory for linear operators as in [6], cannot be applied directly to the Neumann problem since the domain of definition of the Laplace-Neumann operator varies when we consider a family of Riemannian structures on M .

Thanks to Theorem 1.1, it makes sense to get a formula of Hadamard type for the eigenvalues of the Laplace-Neumann operator (cf. Proposition 3.3). In our next result, we show (generically) that all the eigenvalues of the Laplace-Neumann operator are simple. Precisely, we obtain the following.

Theorem 1.2. *Let (M, g_0) be a compact oriented Riemannian manifold and λ an eigenvalue of the Laplace-Neumann operator with multiplicity $m > 1$. Then there exists g in a C^k -neighborhood of g_0 such that the eigenvalues $\lambda(g)$ in a neighborhood of λ are all simple.*

For the next result we consider the set \mathcal{D} of the metrics $g \in \mathcal{M}^k$ such that the eigenvalues of the Laplace-Neumann operator are all simple.

Corollary 1. *The set \mathcal{D} is residual in \mathcal{M}^k .*

To prove her results, Uhlenbeck [15] uses the Thom transversality theorem. In our case, we obtain our results on genericity of the Laplace-Neumann operator by using Hadamard type formulas for eigenvalues (see Proposition 3.3), which at first glance seems to be more appropriated to our situation because possible additional requirements on higher regularity of the Neumann operator family.

It is worth mentioning that the abstract results given by Teytel [14] cannot also be applied to the present case, since the family of operators considered here have variable domain, while that considered by him not.

In the forthcoming work, we intend to present more applications of the method used here. Of particular interest, we intend to study the smooth local structure of the family of metrics on M which preserve the multiplicity of eigenvalues.

2. PRELIMINARIES

Let (M, g) be a compact oriented n -dimensional Riemannian manifold with boundary ∂M . We consider the inner product $\langle T, S \rangle = \text{tr}(TS^*)$ induced by g acting on the space of $(0, 2)$ -tensors on M , where S^* denotes the adjoint tensor of S . Clearly, in local coordinates we have

$$\langle T, S \rangle = g^{ik} g^{jl} T_{ij} S_{kl}.$$

Furthermore, for $f \in C^\infty(M)$ we have $\Delta_g f = \langle \nabla^2 f, g \rangle$ where $\nabla^2 f = \nabla df$ is the Hessian of f . We also recall that each $(0, 2)$ -tensor T on (M, g) can be associated to a unique $(1, 1)$ -tensor by $g(T(X), Y) := T(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. We shall slightly abuse notation here by writing the letter T to indicate this $(1, 1)$ -tensor. So, we can consider the $(0, 1)$ -tensor given by

$$(\text{div} T)(v)(p) = \text{tr}(w \mapsto (\nabla_w T)(v)(p))$$

where $p \in M$ and $v \in T_p M$.

Before deriving our main results we need a lemma which will be crucial in the sequel, and whose proof can be found in Barros-Gomes [1].

Lemma 2.1. *Let T be a symmetric $(0, 2)$ -tensor on a Riemannian manifold (M, g) and φ a smooth function on M . Then we have*

$$\text{div}(T(\varphi Z)) = \varphi \langle \text{div} T, Z \rangle + \varphi \langle \nabla Z, T \rangle + T(\nabla \varphi, Z),$$

for each $Z \in \mathfrak{X}(M)$.

Let now $t \mapsto g(t)$ be a smooth variation of the metric g so that $(M, g(t), \text{dvol}_{g(t)})$ is a Riemannian manifold, where $\text{dvol}_{g(t)}$ is the volume element of $g(t)$ and $\text{d}\sigma_{g(t)}$ is the volume element of $g(t)$ restricted to ∂M . Denote by H a $(0, 2)$ -tensor given by $H_{ij} = \frac{d}{dt} \big|_{t=0} g_{ij}(t)$ and set $h = \langle H, g \rangle$. Similarly \tilde{h} stands for the trace of the $(0, 2)$ tensor \tilde{H} induced by the derivative of $g(t)$ restricted to ∂M . It is easily seen that

$$\frac{d}{dt} \text{dvol}_{g(t)} = \frac{1}{2} h \text{dvol}_g \quad \text{and} \quad \frac{d}{dt} \text{d}\sigma_{g(t)} = \frac{1}{2} \tilde{h} \text{d}\sigma_g.$$

Now, if there is no danger of confusion we will also write $\langle \cdot, \cdot \rangle$ to indicate the metric $g(t)$. Given $X, Y \in \mathfrak{X}(M)$ we can therefore write $X = g^{ij}(t) x_i(t) \partial_j$ and $Y = g^{kl}(t) y_k(t) \partial_l$, with $x_i(t) = \langle X, \partial_i \rangle$ and $y_k(t) = \langle Y, \partial_k \rangle$.

For ease of notation, let $\dot{X} := g^{ij} x'_i \partial_j$ and $\dot{Y} := g^{kl} y'_k \partial_l$, with $x'_i(t) = \frac{d}{dt} x_i$ and $y'_i(t) = \frac{d}{dt} y_i$. Then for every $X, Y \in \mathfrak{X}(M)$ and every $f, l \in C^\infty(M)$, the following properties can be verified:

- (P1) $\frac{d}{dt} \langle X, Y \rangle = -H(X, Y) + \langle \dot{X}, Y \rangle + \langle X, \dot{Y} \rangle$
- (P2) $\frac{d}{dt} \langle \nabla_t f, \nabla_t l \rangle = -H(\nabla f, \nabla l)$
- (P3) $\frac{d}{dt} \langle \nu_t, \nabla_t l(t) \rangle = -H(\nu, \nabla l) + \frac{1}{2} H(\nu, \nu) \langle \nu, \nabla l \rangle + \langle \nu, \nabla l' \rangle,$

where $\nu_t = \frac{\nabla_t f}{|\nabla_t f|}$ and ∇_t means the gradient with respect to $g(t)$. Indeed,

$$\begin{aligned} \frac{d}{dt} \langle X, Y \rangle &= \frac{d}{dt} (g^{ij}(t) x_i(t) y_j(t)) \\ &= -g^{ik} H_{kl} g^{lj} x_i y_j + g^{ij} x'_i y_j + g^{ij} x_i y'_j \\ &= -H(X, Y) + \langle \dot{X}, Y \rangle + \langle X, \dot{Y} \rangle \end{aligned}$$

and this proves (P1). For (P2), observe that if $X = \nabla f$ then $x_i = \langle \nabla f, \partial_i \rangle = \partial_i f$ is independent on t . For (P3), it suffices to note that $\nu_i = \frac{1}{|\nabla f|} \langle \nabla f, \partial_i \rangle$ which implies $\nu'_i = \frac{1}{2} \frac{1}{|\nabla f|} H(\nu, \nu) \partial_i f$ so that $\langle \dot{\nu}, \nabla l \rangle = \frac{1}{2} H(\nu, \nu) \langle \nu, \nabla l \rangle$.

3. INGREDIENTS FOR THE NEUMANN PROBLEM

Throughout this section (M^n, g) stands for an orientable, compact n -dimensional Riemannian manifold, ∂M for its boundary and $t \mapsto g(t)$ is a differentiable variation of g .

Lemma 3.1. *If ν is the exterior normal field at ∂M , then*

$$\left. \frac{d}{dt} \right|_{t=0} \nu(t) = -H(\nu) + \frac{1}{2} H(\nu, \nu) \nu$$

Proof. Let f be a smooth function on M such that $\nu(t) = \frac{\nabla_t f}{|\nabla_t f|}$. Note that

$$\begin{aligned} \frac{d}{dt} \nabla_t f &= -g^{ik} g^{js} H(\partial_k, \partial_s) \partial_i f \partial_j = -g^{js} H(\nabla_t f, \partial_s) \partial_j \\ &= -g^{js} \langle H(\nabla_t f), \partial_s \rangle \partial_j = -H(\nabla_t f). \end{aligned}$$

Hence, we have

$$\left. \frac{d}{dt} \right|_{t=0} \nu(t) = -\frac{1}{2|\nabla f|^3} \langle -H(\nabla f), \nabla f \rangle \nabla f - \frac{1}{|\nabla f|} H(\nabla f) = -H(\nu) + \frac{1}{2} H(\nu, \nu) \nu.$$

□

Our next ingredient is an integral formulae, which provides a generalization of the Equation 3.3 due to Berger [2].

Proposition 3.2. *The following holds for any $f, l \in C^\infty(M)$.*

$$\int_M l \Delta' f \, d\text{vol}_g = \int_M l \left(\frac{1}{2} \langle dh, df \rangle - \langle \text{div} H, df \rangle - \langle H, \nabla^2 f \rangle \right) d\text{vol}_g,$$

where $\Delta' := \left. \frac{d}{dt} \right|_{t=0} \Delta_{g(t)}$.

Proof. By Stokes's Theorem

$$\int_M l \Delta_{g(t)} f \, d\text{vol}_{g(t)} = - \int_M \langle df, dl \rangle d\text{vol}_{g(t)} + \int_{\partial M} l \langle \nu_t, \nabla_t f \rangle d\sigma_{g(t)}.$$

Thus, applying properties (P2) and (P3), at $t = 0$

$$\begin{aligned} \int_M l \Delta' f \, d\text{vol}_g + \frac{1}{2} \int_M l h \Delta f \, d\text{vol}_g &= \int_M H(\nabla f, \nabla l) d\text{vol}_g - \frac{1}{2} \int_M h \langle df, dl \rangle d\text{vol}_g \\ &\quad + \int_{\partial M} l \left(-H(\nu, \nabla f) + \frac{1}{2} H(\nu, \nu) \frac{\partial f}{\partial \nu} \right) d\sigma_g \\ &\quad + \int_{\partial M} l \frac{\tilde{h}}{2} \langle \nu, \nabla f \rangle d\sigma_g. \end{aligned}$$

Rearranging the above equation we have

$$(3.1) \quad \begin{aligned} \int_M l \Delta' f \, d\text{vol}_g &= \int_M H(\nabla f, \nabla l) \, d\text{vol}_g - \int_{\partial M} l H(\nu, \nabla f) \, d\sigma_g \\ &\quad - \frac{1}{2} \int_M (h \langle df, dl \rangle + lh \Delta f) \, d\text{vol}_g + \frac{1}{2} \int_{\partial M} l (\tilde{h} + H(\nu, \nu)) \frac{\partial f}{\partial \nu} \, d\sigma_g. \end{aligned}$$

On the other hand, letting $T = H$, $\varphi = l$ and $Z = \nabla f$ in the Lemma 2.1 we have

$$(3.2) \quad \begin{aligned} \int_{\partial M} l H(\nu, \nabla f) \, d\sigma_g &= \int_M \text{div}(H(l \nabla f)) \, d\text{vol}_g \\ &= \int_M l (\langle \text{div} H, df \rangle + \langle H, \nabla^2 f \rangle) \, d\text{vol}_g + \int_M H(\nabla f, \nabla l) \, d\text{vol}_g. \end{aligned}$$

Moreover,

$$(3.3) \quad \int_{\partial M} lh \langle \nu, \nabla f \rangle \, d\sigma_g = \int_M (lh \Delta f + h \langle df, dl \rangle) \, d\text{vol}_g + \int_M l \langle df, dh \rangle \, d\text{vol}_g.$$

Finally inserting (3.2) and (3.3) in (3.1), we obtain

$$\begin{aligned} \int_M l \Delta' f \, d\text{vol}_g &= \int_M l \left(\frac{1}{2} \langle dh, df \rangle - \langle \text{div} H, df \rangle - \langle H, \nabla^2 f \rangle \right) \, d\text{vol}_g \\ &\quad + \frac{1}{2} \int_{\partial M} l (-h + H(\nu, \nu) + \tilde{h}) \frac{\partial f}{\partial \nu} \, d\sigma_g, \end{aligned}$$

for all $l \in C^\infty(M)$, which is sufficient for completing the proof of proposition, since $\tilde{h} = \text{tr}_g(H|_{\partial M}) = h - H(\nu, \nu)$. \square

In what follows, we obtain Hadamard type formulas for the eigenvalues of the Laplace-Neumann operator. Formal calculations of the Hadamard formula should be required only differentiability of $g(t)$.

Proposition 3.3. *Let $\{\phi_i(t)\} \subset C^\infty(M)$ be a differentiable family of functions and $\lambda(t)$ a differentiable family of real numbers such that $\langle \phi_i(t), \phi_j(t) \rangle_{L^2(M, d\text{vol}_{g(t)})} = \delta_i^j$ for all t and*

$$\begin{cases} -\Delta_{g(t)} \phi_i(t) &= \lambda(t) \phi_i(t) & \text{in } M \\ \frac{\partial}{\partial \nu_t} \phi_i(t) &= 0 & \text{on } \partial M. \end{cases}$$

Then,

$$(3.4) \quad \lambda'(0) \delta_i^j = \int_M \left\langle \frac{1}{4} \Delta(\phi_i \phi_j) g - d\phi_i \otimes d\phi_j, H \right\rangle \, d\text{vol}_g.$$

Proof. Taking the derivative with respect to t at $t = 0$ in both sides of the equation $-\Delta_{g(t)} \phi_i(t) = \lambda(t) \phi_i(t)$, we have $-\Delta' \phi_i - \Delta \phi'_i = \lambda' \phi_i + \lambda \phi'_i$. Thus

$$-\int_M (\phi_j \Delta' \phi_i + \phi_j \Delta \phi'_i) \, d\text{vol}_g = \int_M (\lambda' \phi_j \phi_i - \phi'_i \Delta \phi_j) \, d\text{vol}_g.$$

Now, since $\langle \nu_t, \nabla_t \phi_i(t) \rangle = \frac{\partial}{\partial \nu_t} \phi_i(t) = 0$ on ∂M , we deduce that

$$\langle \nu, \nabla \phi'_i \rangle = H(\nu, \nabla \phi_i) - \frac{1}{2} H(\nu, \nu) \langle \nu, \nabla \phi_i \rangle = H(\nu, \nabla \phi_i) \quad \text{at } t = 0.$$

Moreover, integration by parts gives

$$\begin{aligned}\lambda' \delta_i^j &= - \int_M \phi_j \Delta' \phi_i d\text{vol}_g - \int_{\partial M} \phi_j \frac{\partial}{\partial \nu} \phi_i' d\sigma_g \\ &= - \int_M \phi_j \Delta' \phi_i d\text{vol}_g - \int_{\partial M} \phi_j H(\nu, \nabla \phi_i) d\sigma_g.\end{aligned}$$

Whence,

$$\begin{aligned}-2\lambda' \delta_i^j &= \int_M \phi_j \Delta' \phi_i d\text{vol}_g + \int_M \phi_i \Delta' \phi_j d\text{vol}_g + \int_{\partial M} \phi_i H(\nu, \nabla \phi_j) d\sigma_g \\ &\quad + \int_{\partial M} \phi_j H(\nu, \nabla \phi_i) d\sigma_g \\ &= \int_M \left\langle \frac{1}{2} dh - \text{div} H, \phi_j d\phi_i + \phi_i d\phi_j \right\rangle d\text{vol}_g - \int_M \langle H, \phi_j \nabla^2 \phi_i + \phi_i \nabla^2 \phi_j \rangle d\text{vol}_g \\ &\quad + \int_{\partial M} \phi_i H(\nu, \nabla \phi_j) d\sigma_g + \int_{\partial M} \phi_j H(\nu, \nabla \phi_i) d\sigma_g \\ &= \int_M \left\langle \frac{1}{2} dh, d(\phi_i \phi_j) \right\rangle d\text{vol}_g - \int_M \phi_j (\langle \text{div} H, d\phi_i \rangle + \langle H, \nabla^2 \phi_i \rangle) d\text{vol}_g \\ &\quad + \int_{\partial M} \phi_j H(\nu, \nabla \phi_i) d\sigma_g - \int_M \phi_i (\langle \text{div} H, d\phi_j \rangle + \langle H, \nabla^2 \phi_j \rangle) d\text{vol}_g \\ &\quad + \int_{\partial M} \phi_i H(\nu, \nabla \phi_j) d\sigma_g.\end{aligned}$$

Next use divergence's theorem together with Lemma 2.1 to deduce

$$-2\lambda' \delta_i^j = - \int_M \frac{h}{2} \Delta(\phi_i \phi_j) d\text{vol}_g + 2 \int_M H(\nabla \phi_i, \nabla \phi_j) d\text{vol}_g$$

or equivalently

$$\lambda' \delta_i^j = \int_M \left\langle \frac{1}{4} \Delta(\phi_i \phi_j) g - d\phi_i \otimes d\phi_j, H \right\rangle d\text{vol}_g.$$

□

4. PROOF OF THE MAIN RESULTS

First we deliver the proof of Theorem 1.1, thereby showing that always there exist analytic curves of eigenvalues and eigenvectors for the Laplace-Neumann operator by means of analytic perturbations of the original metric of M . Here, the main technique utilized to do this is the Liapunov-Schmidt method, along the same lines as done by Henry in [4], and continued by Marrocos and Pereira in [8].

Specifically, we consider the Neumann-problem:

$$(4.1) \quad \begin{cases} (\Delta_t + \lambda)u = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu_t} = 0 & \text{on } \partial M, \end{cases}$$

where (M, g_0) is an orientable, compact n -dimensional Riemannian manifold with boundary ∂M , $\Delta_t := \Delta_{g(t)}$, $t \mapsto g(t)$ is an analytic variation of g_0 , with $g(0) = g_0$, and ν_t is a one-parameter family of unit exterior vectors along with $(\partial M, g(t))$.

Proposition 4.1. *Let λ_0 be an eigenvalue of the Laplace-Neumann operator of multiplicity $m > 1$. Then for every $\epsilon > 0$ there is $\delta > 0$ so that for each $|t| < \delta$,*

there exist exactly m eigenvalues (computing their multiplicities) to the problem (4.1) in the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.

Proof. Let $\{\phi_j\}_{j=1}^m$ be an orthonormal basis associated with λ_0 and let

$$Pu = \sum_{j=1}^m \phi_j \int_M \phi_j u \, d\text{vol}_{g_0}$$

be the orthonormal projection on the corresponding eigenspace. As it is well-known, P induces a splitting $L^2(M, d\text{vol}_{g_0}) = \mathcal{R}(P) \oplus \mathcal{N}(P)$ so that, any function u in $L^2(M, d\text{vol}_{g_0})$ can be written as $u = \phi + \psi$, where $\phi \in \mathcal{R}(P) = \ker(\Delta + \lambda_0)$ e $\psi \in \mathcal{N}(P)$.

With this in mind, the Neumann-problem can be equivalently viewed as a system of equations, as to the system:

$$(4.2) \quad \begin{cases} (I - P)(\Delta_t + \lambda)(\phi + \psi) &= 0 & \text{in } M \\ P(\Delta_t + \lambda)(\phi + \psi) &= 0 & \text{in } M \\ \frac{\partial}{\partial \nu_t}(\phi + \psi) &= 0 & \text{on } \partial M. \end{cases}$$

To solve it, we first observe that since ϕ_j and ψ are orthonormal, by the divergence theorem we must have

$$P(\Delta + \lambda)\psi = \sum_{j=1}^m \phi_j \int_M \phi_j (\Delta + \lambda)\psi \, d\text{vol}_{g_0} = \sum_{j=1}^m \phi_j \int_{\partial M} \phi_j \frac{\partial \psi}{\partial \nu} \, d\sigma_{g_0}$$

which implies

$$(\Delta + \lambda)\psi = (I - P)((\Delta + \lambda)\psi) + \sum_{j=1}^m \phi_j \int_{\partial M} \phi_j \frac{\partial \psi}{\partial \nu} \, d\sigma_{g_0}.$$

Thus, we get

$$(\Delta + \lambda)\psi + (I - P)(\Delta_t - \Delta)(\phi + \psi) - \sum_{j=1}^m \phi_j \int_{\partial M} \phi_j \frac{\partial \psi}{\partial \nu} \, d\sigma_{g_0} = 0.$$

Moreover, the part concerning the boundary in (4.2) can be rewritten as:

$$\frac{\partial \psi}{\partial \nu} + \left(\frac{\partial}{\partial \nu_t} - \frac{\partial}{\partial \nu} \right) (\phi + \psi) = 0.$$

Hence solving the first and third equations of (4.2), is equivalent to finding the zeros of the application

$$\begin{aligned} F : \mathbb{R} \times \mathbb{R} \times \mathcal{R}(P) \times H^2(M) \cap \mathcal{N}(P) &\longrightarrow \mathcal{N}(P) \times H^{\frac{3}{2}}(M) \\ (t, \lambda, \phi, \psi) &\mapsto (F_1(t, \lambda, \phi, \psi), F_2(t, \lambda, \phi, \psi)), \end{aligned}$$

where

$$\begin{cases} F_1 = (\Delta + \lambda)\psi + (I - P)(\Delta_t - \Delta)(\phi + \psi) - \sum_{j=1}^m \phi_j \int_{\partial M} \phi_j \frac{\partial \psi}{\partial \nu} \, d\sigma_{g_0} \\ F_2 = \frac{\partial \psi}{\partial \nu} + \left(\frac{\partial}{\partial \nu_t} - \frac{\partial}{\partial \nu} \right) (\phi + \psi). \end{cases}$$

Note that F depends differentially on the variables λ, t, ψ e ϕ . Our intent is to use the Implicit function theorem to show that $F(t, \lambda, \phi, \psi) = (0, 0)$ admits a solution

ψ as function of λ , t and ϕ . To this end, we observe that if $t = 0, \lambda = \lambda_0, \psi = 0$ then

$$(4.3) \quad \frac{\partial F}{\partial \psi}(0, \lambda_0, 0, 0)\dot{\psi} = \left((\Delta + \lambda_0)\dot{\psi} - \sum_{j=1}^m \phi_j \int_{\partial M} \phi_j \frac{\partial \dot{\psi}}{\partial \nu} d\sigma_{g_0}, \frac{\partial \dot{\psi}}{\partial \nu} \right).$$

We claim now that the map given in (4.3) is an isomorphism from $H^2(M) \cap \mathcal{N}(P)$ onto $\mathcal{N}(P) \times H^{\frac{3}{2}}(M)$. Indeed, the proof of this fact can be found in [7].

Hence, by Implicit function theorem there exist positive numbers δ, ϵ and a function $S(t, \lambda)\phi$ of class C^1 at the variables (t, λ) such that for every $|t| < \delta$ and $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $F(t, \lambda, \phi, S(t, \lambda)\phi) = (0, 0)$. Furthermore, $S(t, \lambda)\phi$ is analytic at λ and linear at ϕ . This solves the equation (4.2) in relation to ψ .

We now observe that for every $\phi \in \mathcal{R}(P)$ there exist real numbers c_1, c_2, \dots, c_m so that $\phi = \sum_{j=1}^m c_j \phi_j$. Thus, the second equation in (4.2) can be equivalently seen as a system of equations on the variables c_1, \dots, c_m as below

$$\sum_{j=1}^m c_j \int_M \phi_k(\Delta_t + \lambda)(\phi_j + S(t, \lambda)\phi_j) d\text{vol}_{g_0} = 0, \quad k = 1, 2, \dots, m.$$

In this way, λ is an eigenvalue of Δ_t if and only if $\det A(t, \lambda) = 0$, where $A(t, \lambda)$ is given by

$$A_{kj}(t, \lambda) = \int_M \phi_k(\Delta_t + \lambda)(\phi_j + S(t, \lambda)\phi_j) d\text{vol}_{g_0}.$$

Furthermore, the associated eigenfunctions are given by

$$u(t, \lambda) = \sum_{j=1}^m c_j (\phi_j + S(t, \lambda)\phi_j).$$

In other words, $c = (c_1, \dots, c_m)$ must to satisfy $A(t, \lambda)c = 0$. It turns out that by Rouché theorem, we have that: For every $\epsilon > 0$ there is $\delta > 0$ so that if $|t - t_0| < \delta$, then there exist exactly m -roots of $\det A(t, \lambda) = 0$ in the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. \square

It is worth mentioning that the above proof still does not ensure the existence of an analytic curve of eigenvalues for (4.1). However, the next result goes in this direction.

4.1. Proof of the Theorem 1.1.

Proof. Assume the same conditions of Proposition 4.1. We must show that there exist m -analytic curves of eigenvalues $\lambda_j(t)$ for (4.1) associated to m -analytic curves eigenfunctions $\phi_j(t)$.

The proof strategy is reducing the problem to a finite-dimension analogous one and applying Kato's Selection theorem, see [6]. With this in mind, we will make a slightly different construction than that made in previous proposition.

In order to overcome this drawback, we slightly will change the previous proof of such a way that the new obtained matrix will come to be symmetric.

Let $\{\phi_j\}_{j=1}^m$ be orthonormal eigenfunctions of the Laplace-Neumann associated with λ_0 . For each $j = 1, \dots, m$ consider the following problem:

$$(4.4) \quad \begin{cases} (\Delta + \lambda_0)u &= 0 & \text{in } M \\ \frac{\partial}{\partial \nu_t}(\phi_j + u) &= 0 & \text{on } \partial M \\ Pu = \sum_{j=1}^m \phi_j \int_M \phi_j u d\text{vol}_{g_0} &= 0 & \text{in } M. \end{cases}$$

Consider now the orthogonal complement $[\phi_j]^\perp$ of $\ker(\Delta + \lambda_0)$ in $L^2(M, d\text{vol}_{g_0})$ and define

$$F : (-\delta, \delta) \times H^2(M, d\text{vol}_{g_0}) \longrightarrow [\phi_j]^\perp \times \mathcal{R}(P) \times H^{\frac{3}{2}}(M, d\text{vol}_{g_0})$$

by

$$F(t, w) = ((\Delta + \lambda_0)w, Pw, \frac{\partial}{\partial \nu_t}(\phi_j + w)).$$

Exactly as before we get that $\frac{\partial F}{\partial w}(0, 0)$ is an isomorphism, so by Implicit Function theorem there exist $\delta > 0$ and an analytic function $w_j(t)$ defined on $|t - t_0| < \delta$ such that $F(t, w_j(t)) = 0$. In addition, we obtain for each $|t - t_0| < \delta$ a linearly independent set of functions $\{\varphi_j(t)\}_{j=1}^m$, given by $\varphi_j(t) = \phi_j + w_j(t)$, that satisfy the equation (4.4). By using the Gram-Schmidt orthonormalization process with respect to the inner product

$$(u, v) := \int_M uv d\text{vol}_{g(t)},$$

we can without loss of generality assume that $\{\varphi_j(t)\}_{j=1}^m$ is biorthogonal. Note that the functions $\varphi_j(t)$ belong to $D_t = \{u \in H^2(M, d\text{vol}_{g_0}), \frac{\partial u}{\partial \nu_t} = 0\}$. Moreover, since Δ_t is selfadjoint with respect to the inner product defined above, it follows that the matrix $\int_M \varphi_j \Delta_t \varphi_k d\text{vol}_{g(t)}$ is symmetric.

For a given $T \in \mathcal{S}^{2,k}$, we define now a family of Riemannian metric on M by $g(t) = g_0 + tT$ and let $P(t)$ be given by

$$P(t)u = \sum_{j=1}^m \varphi_j(t) \int_M u \varphi_j(t) d\text{vol}_{g(t)}.$$

We finally define for each $j = 1, \dots, m$,

$$\begin{aligned} G_j : (-\epsilon, \epsilon) \times \mathbb{R} \times H^2(M) &\longrightarrow L^2(M) \times H^{\frac{3}{2}}(M) \times L^2(M) \\ (t, \lambda, w) &\mapsto (G_{j1}(t, \lambda, w), G_{j2}(t, \lambda, w), G_{j3}(t, \lambda, w)) \end{aligned}$$

by

$$\begin{cases} G_{j1} = (I - P(t))((\Delta_t + \lambda))(w + \varphi_j(t)) \\ G_{j2} = \frac{\partial}{\partial \nu_t} w; \\ G_{j3} = P(t)w. \end{cases}$$

Once again, Implicit Function theorem provide a number $\delta > 0$ and functions $w_j(t, \lambda)$ such that for any $|t - t_0| < \delta$ and every $|\lambda - \lambda_0| < \delta$, the equality $G_j(t, \lambda, w_j(t, \lambda)) = (0, 0, 0)$ holds. As we know, λ is an eigenvalue for (4.1) iff there exists a nonzero m -uple $c = (c_1, \dots, c_m)$ of real numbers such that $A(t, \lambda)c = 0$, where

$$(4.5) \quad A_{ij}(t, \lambda) = \int_M \varphi_i(t) (\Delta_t + \lambda) (\varphi_j(t) + w_j(t, \lambda)) d\text{vol}_{g(t)}.$$

That is, λ is an eigenvalue of (4.1) iff $\det A(t, \lambda) = 0$. By Rouché's theorem, there exist m roots near λ_0 counting multiplicity, for each t . So by Puiseux's theorem [16] there exist m -analytic functions $t \rightarrow \lambda_i(t)$ which locally solve the equation $\det A(t, \lambda) = 0$. It can be easily seen that A is symmetric and hence, by Kato's Selection theorem [6], we can find an analytic curve $c^i(t) \in \mathbb{R}^m$ such that $A(t, \lambda_i(t))c^i(t) = 0$, for each $i = 1, \dots, m$. Thus $\psi_i(t) = \sum_{j=1}^m c_j^i(t)(\varphi_j + \omega_j(t, \lambda_i(t)))$ is an analytic curve of eigenfunctions for (4.1) associated with $\lambda_i(t)$. Now, reasoning exactly as Kato in [6, p. 98] we can obtain m -analytic curves of eigenvalues $\{\phi_i(t)\}_{i=1}^m$ such that $\int_M \phi_i(t)\phi_j(t)d\text{vol}_{g(t)} = \delta_i^j$. \square

Remark 1. In the case in which $m = m(\lambda_0) = 1$, the existence of a differentiable curve of eigenvalues through λ_0 follows directly from Implicit Function theorem applied to the map $F : S^k \times H^2(M, d\text{vol}_{g_0}) \times \mathbb{R} \rightarrow L^2(M, d\text{vol}_{g_0}) \times \mathbb{R}$ defined by

$$F(g, u, \lambda) = ((\Delta_g + \lambda)u, \int_M u^2 d\text{vol}_{g_0}).$$

The corresponding formulae to the derivative $\lambda'(t)$ can be obtained by letting $i = j = 1$ in Proposition 3.3.

4.2. Proof of the Theorem 1.2.

Proof. Consider $g(t) = g_0 + tT$, where $T \in \mathcal{S}^{2,k}$ is any covariant symmetric 2-tensor T on M^n so, for t sufficiently small, $g(t)$ is a Riemannian metric. We argue by contradiction: assume λ is an eigenvalue of multiplicity m for the operator Laplace Neumann Δ_{g_0} and, for any $\epsilon > 0$ and for all T , if t is sufficiently small, the operator $\Delta_{g(t)}$ has an eigenvalue $\lambda(t)$ of multiplicity m in $|\lambda(t) - \lambda| < \epsilon$. In this case, by Proposition 4.1, $\lambda(t)$ is only eigenvalues ϵ -close to λ . Since $H = \frac{d}{dt}g(t)\Big|_{t=0} = T$, by Proposition 3.3

$$(4.6) \quad \lambda' \delta_i^j = \int_M \left\langle \frac{1}{4} \Delta(\phi_i \phi_j) g_0 - d\phi_i \otimes d\phi_j, T \right\rangle d\text{vol}_{g_0}.$$

Now consider the symmetrizer tensor

$$S = \frac{d\phi_i \otimes d\phi_j + d\phi_j \otimes d\phi_i}{2}$$

and use the fact $\langle d\phi_i \otimes d\phi_j, T \rangle = \langle d\phi_j \otimes d\phi_i, T \rangle$ to deduce

$$(4.7) \quad \lambda' \delta_i^j = \int_M \left\langle \frac{1}{4} \Delta(\phi_i \phi_j) g_0 - S, T \right\rangle d\text{vol}_{g_0}.$$

If $i \neq j$, we have

$$\int_M \left\langle \frac{1}{4} \Delta(\phi_i \phi_j) g_0 - S, T \right\rangle d\text{vol}_{g_0} = 0,$$

for all $T \in \mathcal{S}^{2,k}$. Thus

$$(4.8) \quad \frac{1}{4} \Delta(\phi_i \phi_j) g_0 = S.$$

By taking the trace in (4.8) we therefore obtain

$$\begin{aligned} \langle \nabla \phi_i, \nabla \phi_j \rangle &= \frac{n}{4} \Delta(\phi_i \phi_j) = \frac{n}{4} (\phi_i \Delta \phi_j + \phi_j \Delta \phi_i + 2 \langle \nabla \phi_i, \nabla \phi_j \rangle) \\ &= \frac{n}{2} (-\lambda \phi_i \phi_j + \langle \nabla \phi_i, \nabla \phi_j \rangle). \end{aligned}$$

Thus, for $n = 2$, it follows from the unique continuation principle [5] that at least one eigenfunction would be vanish, which is a contradiction. Thus $\lambda(t)$ is simple. For $n \geq 3$ we can write

$$(4.9) \quad \frac{n\lambda}{n-2}\phi_i\phi_j = \langle \nabla\phi_i, \nabla\phi_j \rangle.$$

By reasoning exactly as Unlenbeck in [15], for each fixed $p \in M$ we consider α the integral curve in M such that $\alpha(0) = p$ and $\alpha'(s) = \nabla\phi_i(\alpha(s))$. Define $\beta(s) := \phi_j(\alpha(s))$, then

$$\begin{aligned} \beta'(s) &= \langle \nabla\phi_j(\alpha(s)), \alpha'(s) \rangle = \langle \nabla\phi_j, \nabla\phi_i \rangle(\alpha(s)) \\ &= \frac{n\lambda}{n-2}\phi_i\phi_j(\alpha(s)) = \frac{n\lambda}{n-2}\phi_i(\alpha(s))\beta(s) \end{aligned}$$

which is a contradiction since M is compact, thereby proving the theorem. \square

4.3. Proof of the Corollary 1.

Proof. First we note that $\mathcal{D} = \cap_m \mathcal{D}_m$, where

$$\mathcal{D}_m = \{g \in \mathcal{M}^k : \text{the eigenvalues } \lambda \leq m \text{ of } \Delta_g, \text{ are all simple}\}.$$

Hence we need to show that each \mathcal{D}_m are open and dense. The open-ness is quite straightforward, and it follows directly from Proposition 4.1. The denseness is a consequence of Theorem 1.1. \square

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